**1.7.1**. Suppose and and are all real. Show that and and . Show that must be zero if : orthogonal eigenvectors.

**Sol.**

. If , .

**1.7.2**. Which of,,,has two positive eigenvalues? Use a test, don't compute the's. Also find anso that, sois not positive definite.

**Sol**. & , sois not positive definite and does not have two positive eigenvalues.

For example,

& , sois not positive definite and does not have two positive eigenvalues.

& , sois not positive definite (but positive semidefinite) and does not have two positive eigenvalues (but has a positive eigenvalue and a zero eigenvalue).

& , sois positive definite and has two positive eigenvalues.

**1.7.3**. For which numbersandare these matrices positive definite? . With the pivots inand multiplier in, factor eachinto.

**Sol**. ,

,

,

**1.7.4**. Here is a quick "proof" that the eigenvalues of every real matrixare real: False proofgivesso Find the flaw in this reasoning – a hidden assumption that is not justified. You could test those steps on therotation matrix withand.

**Sol**. Flaw: Maybeis complex and.

**1.7.5**. Writeandin the formof the spectral theorem: (keep)

**Sol**.

**1.7.6**. (Recommended) This matrixis antisymmetric and also \_\_\_\_\_. Then all its eigenvalues are pure imaginary and they also have. (for everysofor eigenvectors.) Find all four eigenvalues from the trace of: can only have eigenvaluesor.

**Sol**. This matrixis antisymmetric and also orthonormal.

**1.7.7**. Show that this(symmetric but complex) has only one line of eigenvectors: is not even diagonalizable: eigenvaluesand.is not such a special property for complex matrices. The good property is. Then all eigenvalues are real andhas orthogonal eigenvectors.

**Sol**.

But

**1.7.8**. Thisis nearly symmetric. But its eigenvectors are far from orthogonal: has eigenvectorsand. What is the angle between the eigenvectors?

**Sol**. where

**1.7.9**. Which symmetric matricesare also orthogonal? Thenand.

(a) Show how symmetry and orthogonality lead to.

(b) What are the possible eigenvalues of? Describe all possible.

Thenfor one of those eigenvalue matricesand an orthogonal.

**Sol**. (a)

(b)

**1.7.10**. Ifis symmetric, show thatis also symmetric (take the transpose of). Hereisbyandisby.

Are eigenvalues ofeigenvalues of?

In caseis square and invertible,is called congruent to. They have the same number of positive, negative, and zero eigenvalues: Law of Inertia.

**Sol**. . Sois also symmetric.

. So the eigenvalues are different, unlessis square and orthonormal.

**1.7.11**. Here is a way to show thatis in between the eigenvaluesandof: , is a parabola opening upwards (because of). Show thatis negative at. So the parabola crosses the axis left and right of. It crosses at the two eigenvalues ofso they must enclose. Theeigenvalues ofalways fall between theeigenvalues of. Section 3.2 will explain this interlacing of eigenvalues.

**Sol**.

**1.7.12**. The energycertainly has a saddle point and not a minimum at. What symmetric matrixproduces this energy? What are its eigenvalues?

**Sol**.

**1.7.13**. Test to see ifis positive definite in each case:needs independent columns. and and

**Sol**. is positive definite.

is positive definite.

is positive semidefinite.

**1.7.14**. Find the 3 by 3 matrixand its pivots, rank, eigenvalues, and determinant:

**Sol**. with rank 1.

**1.7.15**. Compute the three upper left determinants ofto establish positive definiteness. Verify that their ratios give the second and third pivots. Pivots = ratios of determinants.

**Sol**. , , .

**1.7.16**. For what numbersandareandpositive definite? Test their 3 determinants: and.

**Sol**. Test for. , ,

Test for. , , no such

**1.7.17**. Find a matrix withandandthat has a negative eigenvalue.

**Sol**. is symmetric about lineand concave upwards.

So in order to have a negative root, it must be, and such as

**1.7.18**. A positive definite matrix cannot have a zero (or even worse, a negative number) on its main diagonal. Show that this matrix fails to have: is not positive when.

**Sol**.

**1.7.19**. A diagonal entryof a symmetric matrix cannot be smaller than all the's. If it were, thenwould have \_\_\_\_\_ eigenvalues and would be positive definite. Buthas a \_\_\_\_\_ on the main diagonal, impossible by Problem 18.

**Sol**. has aon the main diagonal, impossible to be positive definite.

Butmeans positive eigenvalues and positive definite, ifall the.

**1.7.20**. Fromcompute the positive definite symmetric square rootof each matrix.

Check that this square root gives: and

**Sol**.

**1.7.21**. Draw the tilted ellipseand find the half-lengths of its axes from the eigenvalues of the corresponding matrix.

**Sol**.

directionwith half axis, directionwith half axis

**1.7.22**. In the Cholesky factorization, with, the square roots of the pivots are on the diagonal of. Find(upper triangular) forand

**Sol**.

**1.7.23**. Supposeis positive definite (sowhenever) andhas independent columns (sowhenever).

Apply the energy test toto show thatis positive definite: the crucial matrix in engineering.

**Sol**. is positive definite.

**1.7.24**. Forandfind the second derivative matricesand(the Hessian matrices). Test for minimum:is positive definite.is positive definite sois concave up (=convex). Find the minimum point of. Find the saddle point of(look only where first derivatives are zero).

**Sol**. atwhereand

Sois positive semidefinite at the minimum point is at.

atwhereand.

Sois semi negative definite at. No saddle point.

**1.7.25**. Which values ofgives a bowl and whichgives a saddle point for the graph of? Describe this graph at the borderline value of.

**Sol**.

**1.7.26**. Without multiplying, find

(a) the determinant of (b) the eigenvalues of (c) the eigenvectors of (d) a reason whyis symmetric positive definite

**Sol**. , , , , , , positive definite due to positive eigenvalues.

**1.7.27**. For whichandispositive definite? For whichandis it positive semidefinite (this includes definite)? All 5 tests are possible. The energy.

**Sol**. Upper left determinant test. , ,

Ifand, is positive semidefinite.

**1.7.28**. Important! Supposeis positive definite with eigenvalues.

(a) What are the eigenvalues of the matrix? Is it positive semidefinite?

(b) How does it follow thatfor every?

(c) Draw this conclusion: The maximum value ofis.

Note. Another way to 28(c): Maximizesubject to the condition. This leads toand thenand.

**Sol**. (a) where eigenvalues, sois positive semidefinite.

(b)

(c)